QUANTUM ANALYSIS OF THE COMPACTIFICATION PROCESS IN THE MULTIDIMENSIONAL EINSTEIN-YANG-MILLS SYSTEM

O. BERTOLAMI [a]

Departamento de Física, Instituto Superior Técnico,
Av. Rovisco Pais, Lisbon, Portugal

P.D. FONSECA

Department of Physics and Astronomy, Rutgers University,
Piscataway, NJ 08855-0849, U.S.A.

P.V. MONIZ

University of Cambridge, DAMTP, Silver Street,
Cambridge CB3 9EW, England

We study solutions of the Wheeler-DeWitt equation obtained when considering homogeneous and isotropic (up to a gauge transformation) field configurations of the Einstein-Yang-Mills system in $D = 4 + d$ dimensions with an $\mathbf{R} \times S^3 \times S^d$ topology and assuming the Hartle-Hawking boundary conditions.

1 Introduction

We report in this contribution on the result of recent work on the quantum cosmology of the minisuperspace model arising from the coset space dimensional reduction of the D-dimensional Einstein-Yang-Mills (EYM) sytem [1]. Our method consists in exploiting the isometries of an homogeneous and isotropic spacetime in four [2,3,4] or D-dimensions [5] to restrict the possible field configurations. We find that compactifying solutions correspond to maxima of the wave function indicating that these solutions are favoured over the ones where the extra dimensions are not compactified for an expanding Universe. We also find that some features of the wave function of the Universe do depend on the number of extra dimensions [1].

2 Effective Model and Wheeler-DeWitt equation

We consider a $D = d + 4$-dimensional EYM model restricted to spatially homogeneous and (partially) isotropic field configurations, i.e. symmetric fields (up to gauge transformations for the gauge field) under the action of a group $G^{\rm ext} \times G^{\rm int}$ and the gauge group $\hat{K} = SO(N), N \geq 3 + d$. Introducing the most general $SO(4) \times SO(d + 1)$-invariant metric in the space $M^D = M^{4+d} = M^4 \times I^d$ and the $SO(4) \times SO(d + 1)-$ symmetric Ansatz for the gauge field [5] (see also Ref. [2] for a general discussion) into the D-dimensional EYM action leads to the following

[a]Speaker

effective model [5]:

$$S_{\text{eff}} = 16\pi^2 \int dtNa^3 \Bigg\{ -\frac{3}{8\pi k a^2} \left(\left[\frac{\dot{a}}{N}\right]^2 - \frac{1}{4} \right) + \frac{1}{2}\left[\frac{\dot{\psi}}{N}\right]^2 + e^{d\beta\psi}\frac{3}{4e^2a^2}\left(\frac{1}{2}\left[\frac{\dot{f}_0}{N}\right]^2 + \frac{1}{2}\left[\frac{\mathcal{D}_t\mathbf{f}}{N}\right]^2 \right) + e^{-2\beta\psi}\frac{d}{8e^2b_0^2}\left[\frac{\mathcal{D}_t\mathbf{g}}{N}\right]^2 - W(a,\psi,f_0,\mathbf{f},\mathbf{g}) \Bigg\} , \quad (1)$$

where $k = \hat{k}/v_d b_0^d, e^2 = \hat{e}^2/v_d b_0^d, \beta = \sqrt{16\pi k/d(d+2)}, v_d$ is the the volume of S^d for $b = 1$, $\psi = \beta^{-1}\ln(b/b_0)$, b_0 being the equilibrium value of the internal space scale factor, b. Moreover, $\mathcal{D}_t$ denotes the covariant derivative with respect to the remnant $SO(N-3-d)$ gauge field $\hat{B}(t)$: $\mathcal{D}_t\mathbf{f}(t) = \frac{d}{dt}\mathbf{f}(\mathbf{t}) + \hat{B}(t)\mathbf{f}(t)$, and $\mathcal{D}_t\mathbf{g}(t) = \frac{d}{dt}\mathbf{g}(\mathbf{t}) + \hat{B}(t)\mathbf{g}(t)$, such that $\mathbf{f} = \{f_p\}$, $\mathbf{g} = \{g_q\}$ and $\hat{B}$ is an $(N-3-d)\times(N-3-d)$ antisymmetric matrix $\hat{B} = (B_{pq})$. The potential W is on its hand given by:

$$\begin{aligned} W &= e^{-d\beta\psi}\left[-e^{-2\beta\psi}\frac{d(d-1)}{64\pi k b_0^2} + e^{-4\beta\psi}\frac{d(d-1)}{8e^2b_0^4}V_2(\mathbf{g}) + \frac{\Lambda}{8\pi k}\right] \\ &+ e^{-2\beta\psi}\frac{3d}{32e^2(ab_0)^2}(\mathbf{f}\cdot\mathbf{g})^2 + e^{d\beta\psi}\frac{3}{4e^2a^4}V_1(f_0,\mathbf{f}) , \end{aligned} \quad (2)$$

where $\Lambda = v_d b_0^d\hat{\Lambda}$, and $V_1(f_0,\mathbf{f}) = \frac{1}{8}\left[\left(f_0^2+\mathbf{f}^2-1\right)^2 + 4f_0^2\mathbf{f}^2\right]$, $V_2(\mathbf{g}) = \frac{1}{8}\left(\mathbf{g}^2-1\right)^2$ are the contributions associated with the external and internal components of the gauge field, respectively.

Introducing the new variables (μ,ϕ)

$$a = \left(\frac{k}{6\pi}\right)^{\frac{1}{2}} e^{\mu}, \qquad \psi = \left(\frac{3}{4\pi k}\right)^{\frac{1}{2}} \phi ,$$

as well as $\epsilon \equiv \sqrt{12/d(d+2)}$ and turning the canonical conjugate momenta into operators, $\pi_\mu \mapsto -i\frac{\partial}{\partial\mu}$, $\pi_\phi \mapsto -i\frac{\partial}{\partial\phi}$, etc., leads to the Wheeler-DeWitt equation for the wave function $\Psi = \Psi(\mu,\phi,f_0,\mathbf{f},\mathbf{g})$ [1]. To study the compactification process we further set $f_0 = f_0^v,\ \mathbf{f} = \mathbf{f}^v,\ \mathbf{g} = \mathbf{g}^v = \mathbf{0}\ and \mathbf{f}\cdot\mathbf{g} = 0$. From the definitons $v_1 \equiv V_1(f_0^v,\mathbf{f}^v)$, $v_2 \equiv V_2(\mathbf{g}^v) = \frac{1}{8}$ and, after setting $p = 0$ when parametrizating the operator order ambiguity $\pi_\mu^2 \mapsto -\mu^{-p}\frac{\partial}{\partial\mu}\left(\mu^p\frac{\partial}{\partial\mu}\right)$, our problem simplifies to [1]

$$\left[\frac{\partial^2}{\partial\mu^2} - \frac{\partial^2}{\partial\phi^2} + U(\mu,\phi)\right]\Psi(\mu,\phi) = 0, \quad (3)$$

where

$$U(\mu,\phi) = e^{6\mu}\left(\frac{4k}{3}\right)^2\Omega(\mu,\phi) - e^{4\mu} , \quad (4)$$

$$\Omega(\mu,\phi) = -e^{-(d+2)\epsilon\phi}\frac{d(d-1)}{64\pi k b_0^2} + e^{-(d+4)\epsilon\phi}v_2\frac{d(d-1)}{8e^2b_0^4} + e^{d\epsilon\phi-4\mu}v_1\frac{27}{e^2k^2} + \frac{\Lambda}{8\pi k}. \quad (5)$$

3 Solutions with dynamical compactification

Ensuring the stability of compactifation and satifying the observational bound, $|\Lambda^{(4)}| < 10^{-120}\frac{1}{16\pi k}$, requires the cosmological constant to fulfil the condition [5]:

$$\Lambda = \frac{d(d-1)}{16b_0^2} \ , \tag{6}$$

where $b_0^2 = 16\pi k v_2/e^2$. The potential in the minisuperspace simplifies then to

$$U(\mu,\phi) = e^{6\mu - d\epsilon\phi}\frac{2k\Lambda}{9\pi}\left(e^{-2\epsilon\phi}-1\right)^2 - e^{4\mu} + e^{2\mu+d\epsilon\phi}\frac{3\pi}{k}\frac{v_1}{v_2}b_0^2 \ . \tag{7}$$

Next we consider the Hartle-Hawking path integral representation for the ground-state wave function of the Universe [6]

$$\Psi[\mu,\phi] = \int_C D\mu D\phi \exp(-S_{\mathrm{E}}) \ , \tag{8}$$

which allows to evaluate the solution of (49), $\Psi(\mu,\phi)$, close to $\mu = -\infty$ ($S_{\mathrm{E}} = -iS_{\mathrm{eff}}$ and C is a compact manifold with no boundary) and to establish the regions where the wave function behaves as an exponential (quantum regime) or as an oscillation (classical regime). Our results indicate that a generic feature of the wave function is that solutions corresponding to stable compactifying solutions are maxima, meaning that they are indeed the most probable configurations, for an expanding Universe. Moreover, some properties of the wave function were found to depend on the number, d, of internal space dimensions, namely, the regions where the wave function predicts the 4-dimensional metric behaves classically or quantum mechanically (i.e. regions where the metric is Lorentzian or Euclidean) will differ between the $d < 4$ and the $d \geq 4$ cases [1].

We stress that a distinctive feature of our scheme is the non-vanishing contribution of the external components of the gauge field to the potential V_1 in W. It is precisely this feature that allows obtaining a classically stable compactification after inflation [5] and that is responsible for some of the dependence of the wave function in the number, d, of internal dimensions [1]. In this respect our work differs from previous one where stable compactification is achieved through the internal components of a magnetic monopole [7] or via a d-rank antisymmetric tensor [8].

References

1. O. Bertolami, P.D. Fonseca and P.V. Moniz, "Quantum Cosmological Multidimensional Einstein-Yang-Mills Model in a $R \times S^3 \times S^d$ Topology" (gr-qc 9607015), to appear in *Phys. Rev.* D.
2. O. Bertolami, J.M. Mourão, R.F. Picken and I.P. Volobujev, *Int. J. Mod. Phys.* **A6** (1991) 4149.
3. O. Bertolami and J.M. Mourão, *Class. Quantum Gravity* **8** (1991) 1271.
4. O. Bertolami and P.V. Moniz, *Nucl. Phys.* **B439** (1995) 259.
5. O. Bertolami, Yu.A. Kubyshin and J.M. Mourão, *Phys. Rev.* **D45** (1992) 3405.

6. J.B. Hartle and S.W. Hawking, *Phys. Rev.* **D28** (1983) 2960.
7. J.J. Halliwell, *Nucl. Phys.* **B266** (1986) 228.
8. U. Carow-Watamura, T. Inami and S. Watamura, *Class. Quantum Gravity* **4** (1987) 23.